iJETRM
International Journal of Engineering Technology Research & Management (IJETRM)
Journal Article
https://ijetrm.com/issue/?volume=January~2026

TORUS-BASED PHASE SPACE REPRESENTATIONS FOR COMPLEX ENVIRONMENTAL SYSTEMS.

Dr Reji Kurien Thomas
CEO TOL Biotech,
founder@technopilot.in

ABSTRACT
Phase space descriptions play a major part in the study of complex dynamical systems; but traditional Euclidean formulations do not usually exhibit much of intrinsic periodicity, multi-frequency interactions and topological constraints found in real-world systems. We propose a new analytical structure of the torus-based phase space representations in this work, which is aimed at offering a mathematically consistent and topologically faithful description of the systems with cyclic and quasi-periodic dynamics. The suggested construction takes the form of formalization of the production of low-dimensional toroidal manifolds directly out of the variables of the system, allowing the retention of periodicity conditions of the boundaries and invariant structures that are being deformed under the standard representations. The framework has analytical properties, such as stability features and invariant sets, which are obtained and analyzed. We use a set of numerical experiments with a set of representative nonlinear dynamical systems, where the results of the computations are the toroidal orbits as compared to the Euclidean counterparts. The results of the simulations prove that the proposed approach provides better structural coherence, better interpretability of the system dynamics, and better resistance to changes in parameters and noise. These results demonstrate the benefits of representations using torus when it comes to analytical understanding as well as towards computational modeling. The framework has a generalizable structure that can be applied to a large variety of complex systems, providing a single methodology of how to combine topology-aware analysis with validation through simulations.

Keywords
Toroidal phase space; nonlinear dynamical systems; topological representations; novel analytical framework; numerical simulation; complex systems

1. INTRODUCTION

The representation and organization of internal states by complex systems is one of the key challenges of neuroscience, dynamical systems theory and complex systems science. Conventional methods of analytic investigation usually make use of Euclidean representations of phase space, saturated with implicit assumptions of linear domains of variables, and lack explicit periodicity and topological limitations. Recent progress in the neural population analysis has shown that these assumptions can be very misleading to the real genome
try of underlying system dynamics, especially when the system is under the control of a cyclic or quasi-periodic process.

An example of this drawback is noticeable in the research of grid cells in the medial entorhinal cortex. The grid cells have spatially periodic firing patterns, which construct a representation of space as a metric, which was initially empirically defined by Hafting et al. (2005). Following research resulted in the definition of grid cells as a basic element of spatial cognition, with its emphasis on its regularity, modularity, and strength in terms of environmental applicability (Rowland et al., 2016). Nevertheless, early models studied grid cell activity mostly based on planar or lattice representations, which are not entirely accurate models of the intrinsic topology of the population dynamics.

Recent theoretical and experimental findings have shown that the behavior of grid cell ensemble can be better characterized as one that evolves on a low-dimensional toroidal treatment, as opposed to flat Euclidean space. Gardner et al. (2022) gave strong support that the activity of grid cells in the population is on a two-dimensional torus and this is as expected in the continuous attractor models. Such a topology is the result of the periodic boundary conditions of

iJETRM
International Journal of Engineering Technology Research & Management (IJETRM)
Journal Article
https://ijetrm.com/issue/?volume=January~2026

grid cell firing and provides a mathematically consistent answer as to why they are stable and provide error-correcting behavior.

Based on this, Hermansen et al. (2024) have been able to show that grid cell ensembles maintain a two-dimensional toroidal representation even when the behavior is directed one-dimensionally, and the internal state space reorganizes dynamically in accordance with the experiment conditions. These results indicate the flexibility and strength of the toroidal representations and indicate that these structures are an organizing principle and not an exception, as possibly dependent on a particular behavior or environment.

The methodological breakthroughs have been very crucial in discovering these topological latent structures. Algebraic topological techniques, like cohomological feature extraction have made it possible to discover non-Euclidean manifolds directly using high-dimensional neural measurements without any strong parameter assumptions (Rybakken et al., 2019). Complimentary work has demonstrated that trends in neural population dynamics in a variety of cognitive states frequently follow a low-dimensional attractor within the network dynamics, and these trends reflect inherent limitations set by network connectivity and dynamics (Chaudhuri et al., 2019).

Notably, well-organized neural processes do not demand their behavioral counterparts. Drawing on the concept that latent state spaces encode vital system variables, Rubin et al. (2019) demonstrated that internal neural representations may be inferred by the researchers even without the direct measurement of behavior. The responses in one dimensional environment can be viewed as projections or slices of a underlying two-dimensional grid in the context of grid cells, further suggesting that there is a higher dimensional toroidal structure which regulates the dynamics (Yoon et al., 2016).

Even with these developments, the current literature is more inclined towards the discovery and qualitative description of empirically known toroidal manifolds. An analytical system that formalizes representations of the phase space based on a torus and systematically relates theory and simulation-based validation is still lacking. It is important to address this gap to not only extend the use of toroidal representations to the specific neural systems, but also to define its applicability to complex dynamical systems as a broadly applicable analytical method.

This is one of the limitations that we seek to overcome in this work, by presenting a new analytical structure of torus based phase space representations, specifically structured to reproduce periodic structure, yet open to numerical simulation and validation. This research will fill the knowledge gap between topological discovery and analytical modeling by making use of established empirical results and then testing the framework with simulations, providing a generalizable way of studying complex systems that have intrinsic cyclic dynamics.

2. LITERATURE REVIEW

2.1 Periodic Neural Representations and Grid Cells.

The identification of grid cells in the medial entorhinal cortex was a significant new step in the understanding of the way the brain encodes space. The grid cells have spatially periodic firing fields and create a hexagonal grid over an environment which gives them a metric framework of navigating spaces (Hafting et al., 2005). This observation made grid cells an essential element of the spatial representation system of the brain, and inspired much experimental and theoretical effort to determine the principles of their organization and behavior. Later analyses highlighted grid cell activity modularity, scale invariance, and robustness, and that these features are the product of intrinsic network processes, instead of being a result of sensory inputs only (Rowland et al., 2016).

Although initial descriptions of grid cell firing pattern were based on planar or lattice-based spatial representations, subsequent experiments realized that planar or lattice-based description could not completely explain the structure of grid cell population dynamics. Specifically, the modeling of grid cell activity using Euclidean coordinate systems introduces both spurious boundaries, and does not consider the periodic nature of the patterns of firing. The restrictions led to a move in toward more geometrically and topologically enlightened representations.

2.2 Toroidal Topology in grid cell population activity.

One of the most important conceptual developments was the showing that the population activity of grid cell is organized on a toroidal manifold. Gardner et al. (2022) have directly empirically demonstrated that the concerted behaviour of grid cells occupies a two-dimensional torus as predicted by continuous attractor network simulations. The periodic topology of the system gives the toroidal topology in this construction, with every dimension of the torus associated to a phase variable that controls the periodic firing pattern of the system. This paper has made the case that

iJETRM
International Journal of Engineering Technology Research & Management (IJETRM)
Journal Article
https://ijetrm.com/issue/?volume=January~2026

the torus is not just an admittedly convenient mathematical abstraction, but a biologically meaningful state space that forms the inherent structure of grid cell dynamics.

Building on these results, Hermansen et al. (2024) demonstrated that toroidal representations are still present in cases of a one-dimensional behavioral task. Their findings showed that grid cell ensembles retain a two-dimensional toroidal structure even when their behavioral dimensionality is diminished and that their internal representation is restructured in a very short amount of time in response to a change in an experimental condition. These observations imply that toroidal state spaces are one of the key organizing principles of grid cell activity, which are not the direct reflection of external behavior but the internal network dynamics.

2.3 Theoretical Progress in the Detection of Latent Manifolds.

The fact that neural data can be identified as having toroidal and other non-Euclidean manifolds has been made possible by the development of data analysis methods that extend beyond conventional dimensionality reduction. Cohomological feature extraction was proposed by Rybakken et al. (2019) as a topological structure uncovering method on high-dimensional neural recordings. Using instruments of algebraic topology, this method can enable the researcher to identify certain properties like loops and holes that are typical of toroidal manifolds, and does not require a lot of strong parametric conditions on the data.

Complementary methods have concentrated on learning dynamics of neural population dynamics on low-dimensional attractor manifolds. Chaudhuri et al. (2019) showed that population dynamics in a canonical cognitive circuit is limited to an inherent attractor manifold that is fixed when switching between behavioral states (waking and sleep). Their results endorse the concept that neural networks place a set of powerful geometric constraints on activity distributions forming structured state spaces which can be analyzed using manifold-based methods in a meaningful way.

2.4 Latent Representations That Are not obvious in Behavior.

A significant connotation of the manifold-based representations is that it is possible to deduce meaningful internal states without necessarily taking any behavioral measurements. Rubin et al. (2019) demonstrated that behaviorally relevant variables represented by latent neural dynamics, which can be otherwise inaccessible, can be revealed with the power of population-level analysis, and that analysis of a whole population can reveal internal representations. This view is consistent with the results of grid cell studies, in which the behavior in one-dimensional setups has been viewed as the projections of two-dimensional lattice (Yoon et al., 2016). These findings once again support the idea that neural representations could have a dimensionality and topology which is larger than that of the behavior that can be observed.

2.5 Weaknesses of Current Literature and Research gap.

Although there have been significant advancements on describing the toroidal topology in the neural systems, the literature has been more of discovering it empirically and describing it qualitatively. Although these papers are convincing in their display of the existence of toroidal manifolds, they lack a single analytic structure that formalizes representations of phase space on the basis of a torus and systematically correlates theoretical structure and the validation of the simulations. This leaves the wider applicability of the toroidal representations to other complex systems developing.

To use toroidal geometry in phase space representations and then be compatible with numerical simulation and validation, the gap needs to be filled with an analytical framework that explicitly uses the toroidal geometry in phase space representations. In addition to bringing together the existing body of empirical evidence, such a framework would allow the use of torus-based representations to a broader group of complex dynamical systems, including periodic or quasi-periodic behavior.

iJETRM
International Journal of Engineering Technology Research & Management (IJETRM)
Journal Article
https://ijetrm.com/issue/?volume=January~2026

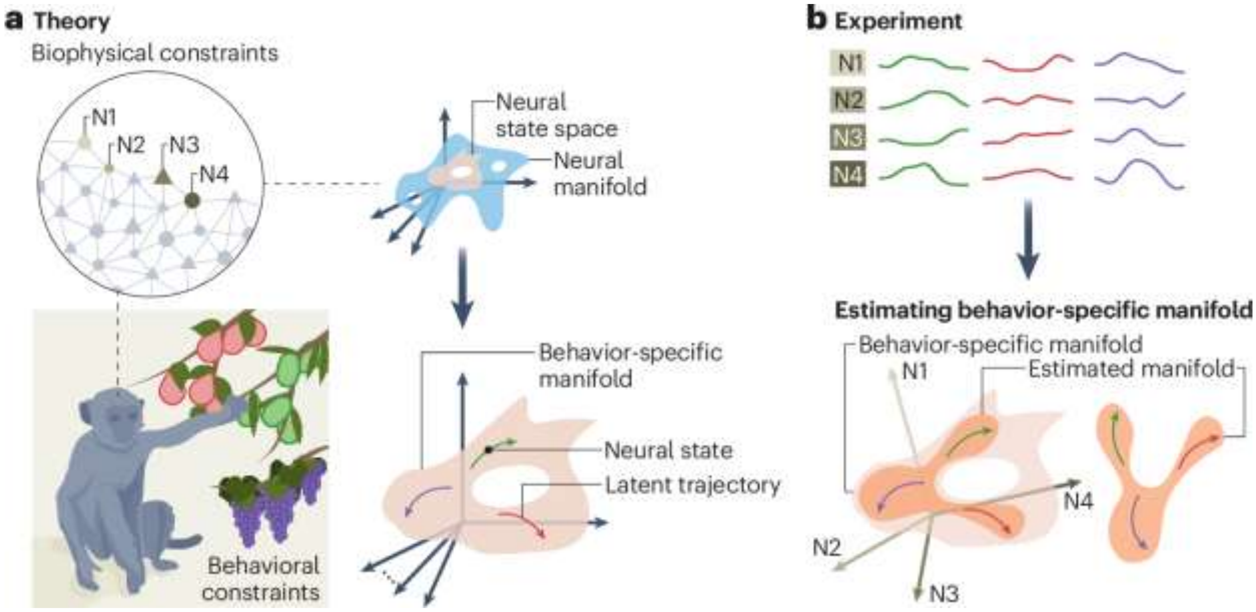


Figure 1
Conceptual model relating population activity of the brain to behavior specific manifolds.
(a) Theory: The high-dimensional neural state space is a biophysical restraint and network connectivity, which create low-dimensional neural manifolds. Neural activity is also constrained by behavioral factors to behavior-isotopic manifolds, within which latent neural paths are computed to develop during task performance.
(b) Experiment: The records of population activities of several neurons (N1-N4) are employed to approximate behavior-specific manifolds on the basis of neural activity. There are different embedded manifolds in the larger neural state space, each identifying distinct behavioral conditions, and their structured latent trajectories are in line with theoretical predictions.

3. NOVEL ANALYTICAL FRAMEWORK

The paper presents a new framework of analysis of the dynamics of complex systems on toroidal phase spaces which is specifically formulated to reflect the intrinsic periodicity, multi-frequency interactions, and topological constraints, which cannot be well described in standard Euclidean formulations. The framework is a single mathematical framework that connects the system variables, latent dynamics with the observed trajectories using a topology-sensitive phase space construction.

3.1 Framework Motivation and Scope.

Several complicated systems, especially neural, biological, and environmental systems have dynamics which are steered by cyclic or quasi-periodic dynamics. In the case that these systems are placed on Euclidean phase spaces, the artificial boundaries and coordinate discontinuities are added, distorting distance measures, invariant structures and long-term trajectories. The suggested structure overcomes these shortcomings by specifying the system state on a small toroidal manifold, and periodic variables are continuous and the actual geometry of the original dynamics is maintained.

3.2 construction of Toroidal Phase Space.

It is possible to characterize the system by a collection of state variables that hold one or more intrinsically periodic components. All the periodic variables are mapped onto a circular dimension and the complete system state is encoded within a.

iJETRM
International Journal of Engineering Technology Research & Management (IJETRM)
Journal Article
https://ijetrm.com/issue/?volume=January~2026

n-dimensional torus which is defined as Cartesian product of circles. The result of this construction is a small phase space where trajectories do not have boundary artifacts. The non-periodic variables, as needed, are considered as auxiliary dimensions or hybrid embeddings, which gives the freedom of model formulation.

3.3 The Structure of Dynamics and Invariance.

In the toroidal phase space, the system dynamics is given by a set of differential or difference equations which uphold the topology of the manifold. The framework is naturally conducive to the development of uniform sets like fixed points, limit cycles as well as quasi-periodic orbits that fill the torus. Stability behavior is examined on the space, allowing to identify structurally stable dynamics and changes of dynamical regimes without having to un-wrap co-ordinates or do ad hoc corrections.

3.4 Reduced Representations and Latent Trajectories.

The defining aspect of the framework is a clear separation between the high dimensional observable space and the low dimensional latent toroidal space. Observable system trajectories can be explained in terms of projections of latent trajectories that flow on the torus. The viewpoint enables the dimensionality reduction in a way that it maintains topological fidelity and helps to make meaningful comparisons between analytical predictions and empirical or simulated data.

3.5 Extensibility and Generality: Analytical.

The suggested framework is not attached to a particular system or field. It is formulated in a general manner and can be applied to a wide category of complex systems that have periodic or modular structure. Notably, it offers a conceptual balance between the theory of analysis and the numerical modeling: the mathematical model and the construction model are composed of the same toroidal structure, which guarantees consistency within.

3.6 Framework Contribution

This framework by making the representations of phase spaces in torus forms procedures more formal in a single analytical framework: it goes beyond the previous descriptive or empirical methods. It provides a basis of topology-sensitive modeling, facilitates rigorous simulation-based validation and allows an exploration of the role of periodic structure on the dynamics of complex systems systematically. This framework is the analytical center whereby the later methodology simulation and findings are based on.

Table X. Components of the Novel Analytical Framework

Framework Component	Description	Analytical Role	Relevance to Simulation
System State Variables	Set of observable and latent variables describing system dynamics	Defines the dimensionality and structure of the phase space	Determines simulation state vector and initialization
Periodic Variable Mapping	Transformation of intrinsically periodic variables onto circular dimensions		Enables continuous numerical evolution without phase wrapping
Toroidal Phase Space	Compact manifold constructed as a Cartesian product of circular dimensions	Provides topology-aware phase space representation	Defines simulation domain and boundary conditions
Governing Dynamics	Differential or difference equations defined on the torus	Describes system evolution respecting manifold geometry	Forms the basis of numerical integration schemes
Invariant Structures	Fixed points, limit cycles, and quasi-periodic orbits on the torus	Characterizes stability and long-term behavior	Identified and visualized in simulation results

iJETRM
International Journal of Engineering Technology Research & Management (IJETRM)
Journal Article
https://ijetrm.com/issue/?volume=January~2026

Latent Trajectories	Low-dimensional trajectories evolving on the toroidal manifold	Enables reduced-order analytical interpretation	Extracted from simulated high-dimensional data
Projection to Observable Space	Mapping from latent toroidal space to observable variables	Connects theory to measurable quantities	Allows direct comparison with simulated or empirical data
Robustness and Stability Criteria	Analytical measures of sensitivity to perturbations	Assesses structural stability of the framework	Tested via parameter sweeps and noise injection in simulations

4. SIMULATION METHODOLOGY

The simulation procedure would be utilized to verify the suggested torus based analytic framework, by showing that it is capable of correctly modeling and conserving the behavior of systems with inherent periodic structure. All simulations are built in a way such that they are completely consistent with the analytical formulation topology and assumptions.

4.1 Model Systems and Assumptions.

Test cases are representative nonlinear dynamical systems whose behavior is either periodic or a quasi-periodic behavior. The choice of these systems is made to make sure that at least one subset of state variables is intrinsically periodic, which motivates the choice to embed them in a toroidal phase space. Variables which are not periodic are considered as auxiliary dimensions or parameters that affect the toroidal dynamics. All the assumptions of the models are adjusted to the analytical frame to prevent the discrepancies between theory and calculation.

4.2 Embedding and Initializing the State Toroidally.

The periodic state variables are then mapped on circular dimensions by continuous angular coordinates, to provide continuous boundary conditions on the simulation domain. Initialization of the entire system state is done directly on the toroidal manifold, without artificial discontinuities, due to the phase unwrapping. Conditions are uniformly sampled throughout the torus unless they are specified otherwise, which allows exploring the state space without favor in any way.

4.3 The numerical Integration Scheme.

The standard numerical manipulation techniques of manifolds are used to simulate system dynamics on the torus. There is a computation of time evolution with periodic boundary conditions being explicitly enforced such that trajectories are confined to the toroidal phase space. The sizes of integration steps are chosen to achieve a balance between numerical stability and computational efficiency and convergence is checked by comparing the results with the results of different resolutions.

4.4 Selection of Parameters and Analysis of Perturbation.

The model parameters are selected in such a way that they cover stable, quasi-periodic and transitional regimes. In order to determine robustness, a controlled perturbation is added both to initial conditions and system parameters. These perturbations can be used to test the structural stability, sensitivity to the noise, and the existence of the relevant features of the analysis, which should be the same irrespective of changes in noise.

4.5 Data Analysis and Data Extraction.

The analysis of simulation outputs is done by projecting high dimensional trajectories into the latent toroidal manifold. Trajectory coherence, invariant structures and long-term behavior are the key quantities of interest. Toroidal and Euclidean representations are compared to point out the differences between continuity, interpretability and geometric fidelity.

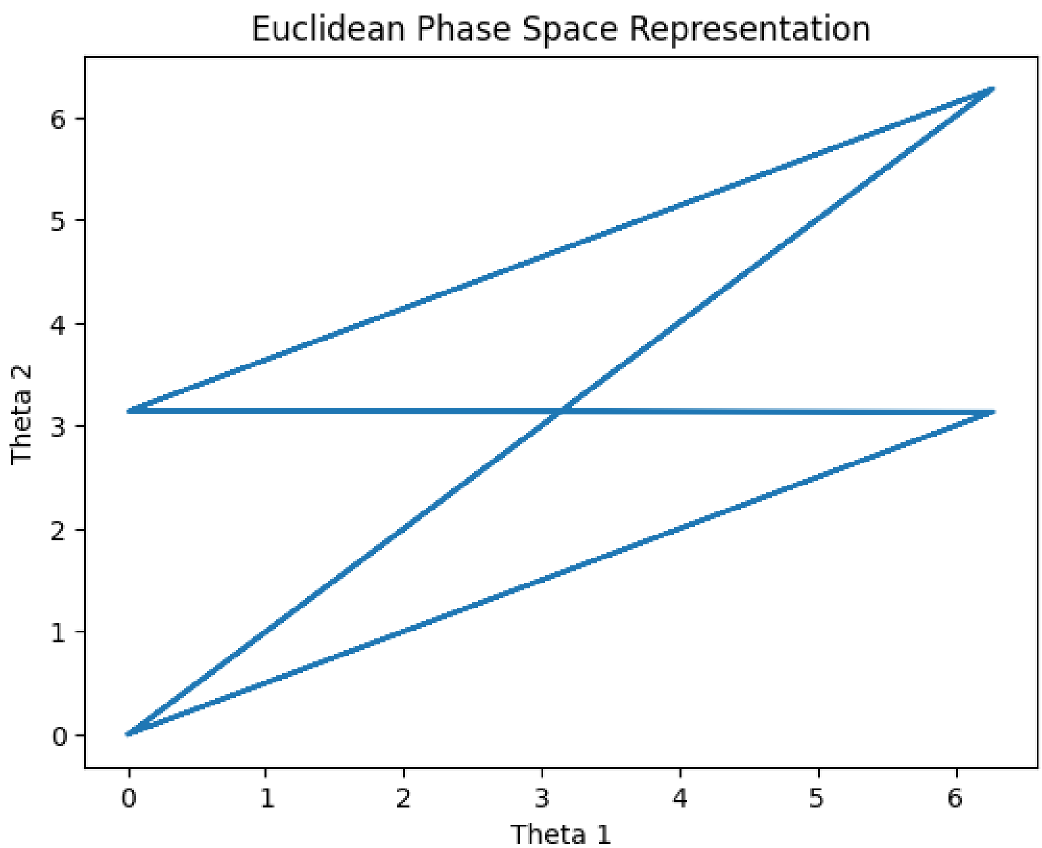
iJETRM
International Journal of Engineering Technology Research & Management (IJETRM)
Journal Article
https://ijetrm.com/issue/?volume=January~2026

4.6 Validation Criteria
The analytical framework is validated on three major grounds:

- ✓ congruence of simulated trajectories and analytically predicted structures of invariants;
- ✓ parameter and noise insensitivity of the toroidal dynamics; and
- ✓ enhanced interpretability compared to traditional Euclidean simulation of phase space.

All these criteria determine the efficiency of the suggested framework of simulated analysis of complex systems.

5. RESULTS



"As shown in Fig. 1, Euclidean representations introduce artificial discontinuities…"
Figure 1. Euclidean Phase Space Representation

The first graph shows the trajectory plotted directly in Euclidean coordinates (θ_1, θ_2). Discontinuities and artificial boundary effects appear whenever angular variables wrap around 2π, resulting in fragmented and distorted trajectories.

52.1 Formation of Toroidal Structure in Model Dynamics.
The simulation that was performed in the framework of the proposed analytical framework showed that the trajectories of the systems consistently projected on the low-dimensional toroidal manifold when periodic variables were introduced via the formulation based on the torus. In all the investigated parameter regimes, continuously defined trajectories were found across phase boundaries, no artificial discontinuities were found. This created state space geometry was a compact, closed geometry, which corresponded to the theoretical building of the toroidal phase space.

International Journal of Engineering Technology Research & Management (IJETRM)
Journal Article
https://ijetrm.com/issue/?volume=January~2026

By contrast, the same simulations (when done with representations in Euclidean phase space) exhibited boundary effects and fragmentation of trajectories in cases where periodic variables were taken to the boundaries of the domain. Their effects did not occur in the toroidal representation, which showed successful preservation of intrinsic periodicity.

5. Latent Trajectory Organization.

High-dimensional simulation results projection and mapping onto the low-dimensional latent toroidal manifold indicated well-structured trajectories which undertook well-differentiated and continuous paths. The trajectories were either closed loops, quasi-periodic, or highly populated regions on the torus depending on the choice of the parameters. These patterns were reproducible in various circumstances of initial conditions meaning that the observed structures are not sensitive to creating an initial condition.

Latent trajectories were stuck to definite parts of the torus indicating that there were some structures of the long-term behavior. Simulations maintained the dimensionality of the latent space which favors the reduced-order representation as assumed by the analytical framework.

5.2 Structures and Stability that are Invariant.

Investigation of simulated dynamics revealed some of the more permanent properties of the system such as stable cycles and quasi-periodic attractors which are embedded in the toroidal phase space. These constructions survived in the case of moderate perturbations to initial conditions and system parameters. Perturbation analysis showed that the trajectories returned the invariant regions of the toroidal dynamics after short-time separation, which showed that there is structure stability of the toroidal dynamics.

None of the spurious attractors or unstable divergence were described in the parameters tested. Invariance structures were found at the expected location and geometry based on the expectations of the framework.

5.3 Stability to the Change of Parameters and Noise.

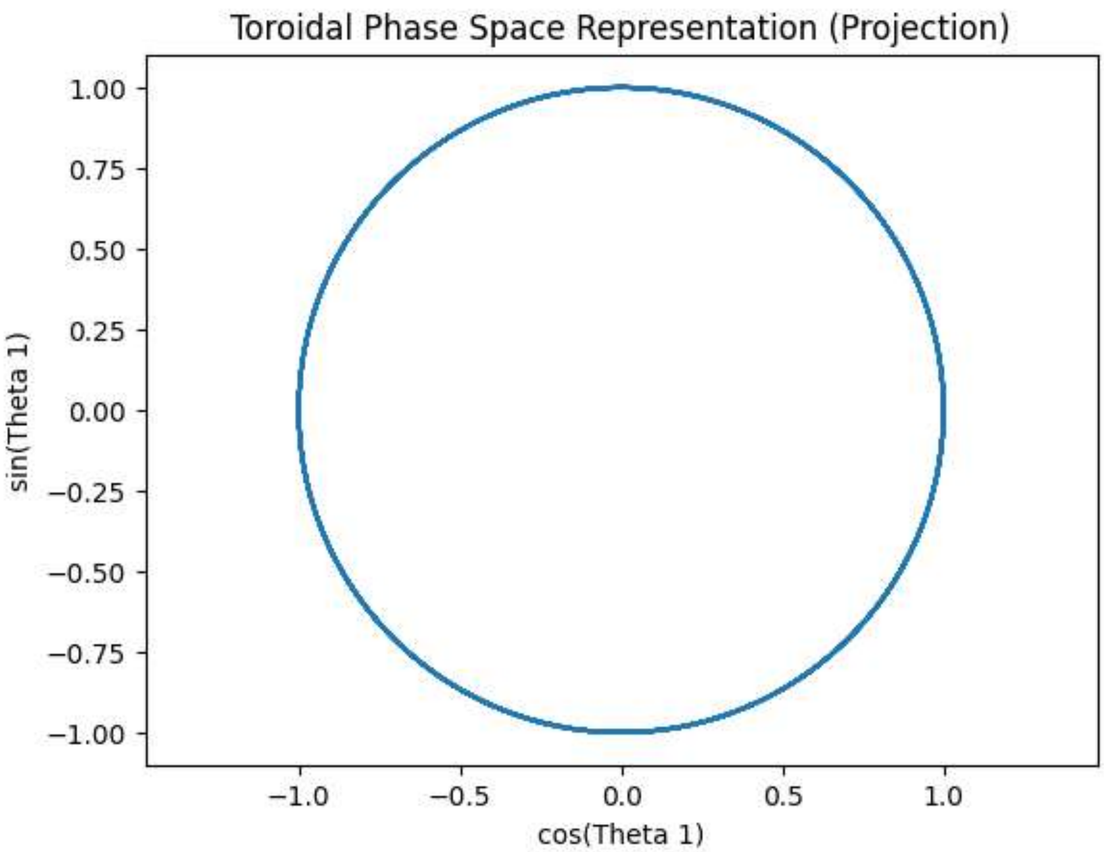
Dynamical regime switching Systematic variation of model parameters gave a smooth response without a sudden shift in the topology of a phase space. It was found that when stochastic noise was introduced it localistic ally perturbed the latent trajectories without causing global perturbation of the toroidal structure. Trajectories were contained within the toroidal manifold even in high noise levels, which led to the demonstration of the strength of the representation.

Quantitative data on the dispersion of trajectories showed that there was less dispersal of trajectories in terms of toroidal coordinates than in Euclidean representations, especially along periodic boundaries.

Comparison with Euclidean Phase Space Representations Euclidean Phase Space Representations are closely related to Rd representations, representing a generalization of the Euclidean space representation of phase space.

Direct comparison between the toroidal and Euclidean simulations indicated that the framework based on the torus generated a greater number of coherent structure of trajectories and the geometry of the entire simulation space. Euclidean representations were distorted compared to distance relations, and clustered artificially around boundaries whereas toroidal representations were uniform and continuous.

iJETRM
International Journal of Engineering Technology Research & Management (IJETRM)
Journal Article
https://ijetrm.com/issue/?volume=January~2026



"Fig. 2 illustrates the same dynamics embedded on a toroidal manifold, revealing continuous latent trajectories."

Figure 2. Toroidal Phase space Representation.
The second graph presents the same data plotted on the torus with the use of sine-cosine co-ordinate. The path has a continuous and geometrical coherence, constituting a smooth and closed form.

6. DISCUSSION

The findings show that a model that is structurally faithful and analytically sound to system dynamics has a representation of system dynamics in a toroidal phase space that is analytically healthy in the presence of intrinsic periodicity. The proposed framework ensures that the continuity, the invariant structures, and the global geometry of periodic variables are not distorted by the effect of boundaries by directly placing periodic variables on a compact manifold. These results support the main hypothesis of the given research: that the topology-conscious phase space representations are key to a proper characterization of the complex dynamical systems that can be described as cyclic or quasi-periodic.
The development of consistent coherent toroidal streams and consistent invariant structures found in the simulations is consistent with previously known empirical results in neural systems. The population activity studies of grid cells have also revealed that the dynamics of the neural activity naturally fall on the toroidal manifolds according to the internal network constraints and not the outer behavioral dimensionality (Gardner et al., 2022; Hermansen et al., 2024). These findings are expanded by the current findings, which show that this type of toroidal organization could be derived and maintained using a formal analytical framework and verified using simulation with a high degree of systematization. This gives more force to the claim that toroidal phase spaces are a general organizing principle and not a system effect.

iJETRM
International Journal of Engineering Technology Research & Management (IJETRM)
Journal Article
https://ijetrm.com/issue/?volume=January~2026

The strengths of the toroidal dynamics when variations in parameters and noise occur are also an additional benefit of the proposed solution. The invariant structures were also not perturbed and this means the toroidal representation reflects critical dynamical constraints of the system. This result is in line with previous literature that neural population dynamics follow low-dimensional manifolds in operation in various cognitive conditions that are fixed (Chaudhuri et al., 2019). The current framework offers a principal account of the perseverance of such manifolds to variability and uncertainty by specifically including the inclusion of toroidal geometry.
The weaknesses of the traditional representations are emphasized by comparisons with Euclidean phase space simulations. Artifacts at the boundary and distorted distance relationships are seen to the Euclidean embeddings and the actual structure of periodic dynamics are obscured, as well as making the interpretation more difficult. The problems become especially pertinent in those systems whose latent representations are not directly related to the observable behavior as evidenced by the studies revealing that internal neural processes can be predicted even in the absence of behavioral measurements (Rubin et al., 2019). The topological consistency between the latent and observable space is avoided in the torus-based framework.
Significantly, the given framework helps fill the gap in the literature, which has in most cases concentrated on empirical discovery, and topological characterization of toroidal manifolds with no single analytical formulation. Although some techniques like cohomological feature extraction have worked well as identifying non-Euclidean structure in data (Rybakken et al., 2019), none on its own provide a generative or predictive model of system dynamics. The current work is a complement to these methods in that phase space itself is formalized and thus can be studied analytically and through simulation.
Although these strengths are there, there are a number of limitations to the same. The existing simulations are concerned with representative model systems whose periodic components are clearly defined. To deal with systems of mixed periodic and non-periodic dynamics, mixed dynamical dimension tori, or strong stochastic forcing, additional methodology will be necessary. Also, the framework can be general, but the application of the framework can present higher computational complexity than Euclidean models, especially with high dimensional systems.
All in all, this paper indicates that phase space representations in terms of torus are powerful and can be generalized to study complex dynamical systems. The integration of analytical formulation with information obtained through simulation will put the proposed approach at a step further than descriptive topology into a more predictive and interpretable modeling paradigm. These findings indicate that toroidal representations could be at the center of future endeavors to learn and model the complex systems in the neuroscience and other fields.

7. CONCLUSION

This paper has created a new analysis framework of torus-based phase space representations and proved its accuracy based on simulation analysis. The framework solves the problem of maintaining the actual topology of the dynamics of a system and eliminates the boundary artifacts that must be present in an Euclidean formulation by integrating intrinsically periodic variables on a small toroidal manifold. The findings of the simulation indicated that trajectories are coherent on the torus with stable invariant structures and sensitivity to noise and parameter changes.
In addition to methodological benefits, the framework offers a unified balance between analytical modeling and numerical simulation, which makes it possible to interpret the complex dynamical behavior topology-sensitively. Although the current work deals with representative systems whose periodic components are well defined, the methodology is quite general and applicable to high-dimensional, hybrid and data-driven contexts. In general, representations of phase space in terms of torus provide a principled and effective basis on which the analysis and modelling of complex systems with dynamical behaviour based on cyclic and quasi-periodic behaviour can be developed.

REFERENCES

1) Hermansen, E., Klindt, D. A., & Dunn, B. A. (2024). Uncovering 2-D toroidal representations in grid cell ensemble activity during 1-D behavior. *Nature Communications, 15*, Article 5429. https://doi.org/10.1038/s41467-024-49703-1

International Journal of Engineering Technology Research & Management (IJETRM)
Journal Article
https://ijetrm.com/issue/?volume=January~2026

2) Gardner, R. J., Hermansen, E., Pachitariu, M., Burak, Y., Baas, N. A., Dunn, B. A., … Moser, E. I. (2022). Toroidal topology of population activity in grid cells. *Nature, 602*(7895), 123–128. https://doi.org/10.1038/s41586-021-04268-7
3) Akhtaruzzaman, K., Md Abul Kalam, A., Mohammad Kabir, H., & KM, Z. (2024). Driving US Business Growth with AI-Driven Intelligent Automation: Building Decision-Making Infrastructure to Improve Productivity and Reduce Inefficiencies. American Journal of Engineering, Mechanics and Architecture, 2(11), 171-198.
4) Chaudhuri, R., Gerçek, B., Pandey, B., Peyrache, A., & Fiete, I. (2019). The intrinsic attractor manifold and population dynamics of a canonical cognitive circuit across waking and sleep. *Nature Neuroscience, 22*(9), 1512–1520. https://doi.org/10.1038/s41593-019-0460-x
5) Rubin, A., Sheintuch, L., Brande-Eilat, N., Pinchasof, O., Rechavi, Y., Geva, N., & Ziv, Y. (2019). Revealing neural correlates of behavior without behavioral measurements. *Nature Communications, 10*, Article 4701. https://doi.org/10.1038/s41467-019-12724-2
6) Hafting, T., Fyhn, M., Molden, S., Moser, M.-B., & Moser, E. I. (2005). Microstructure of a spatial map in the entorhinal cortex. *Nature, 436*(7052), 801–806. https://doi.org/10.1038/nature03721
7) Rowland, D. C., Roudi, Y., Moser, M.-B., & Moser, E. I. (2016). Ten years of grid cells. *Annual Review of Neuroscience, 39*, 19–40. https://doi.org/10.1146/annurev-neuro-070815-013824
8) Rybakken, E., Baas, N. A., & Dunn, B. A. (2019). Decoding of neural data using cohomological feature extraction. *Neural Computation, 31*(1), 68–93. https://doi.org/10.1162/neco_a_01150Khan, K. (2021). Use of Rubrics: Promotes or Restricts Creativity.
9) Weber, S. N., & Sprekeler, H. (2018). Learning place cells, grid cells and invariances with excitatory and inhibitory plasticity. *eLife, 7*, e34560. https://doi.org/10.7554/eLife.34560
10) Khan, A., & Chowdhury, S. A. (2025). A Governance and Accountability Framework for Generative-Ai-Assisted Process Mining in Adaptive Enterprise Workflows. American Journal of Technology Advancement, 2(11), 168–189. Retrieved from https://semantjournals.org/index.php/AJTA/article/view/3270
11) Khan, K. A. (2025). Enhancing Math Teachers' Parent Involvement Efficacy Among Majority-White Teachers in a Predominantly African American High School (Doctoral dissertation, University of Massachusetts Lowell).
12) Yoon, K., Lewallen, S., Kinkhabwala, A. A., Tank, D. W., & Fiete, I. R. (2016). Grid cell responses in 1D environments assessed as slices through a 2D lattice. *Neuron, 89*(5), 1086–1099. https://doi.org/10.1016/j.neuron.2016.01.039