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# A QUICK AND EFFICIENT NUMERICAL TECHNIQUE FOR SOLVING SECOND ORDER NONLINEAR TWO-POINT BOUNDARY VALUE PROBLEMS

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**Abstract:**
The present study is relevant to solve second order nonlinear second order two-point boundary value problems by using fast, rapid and efficient numerical schemes named as Adams numerical scheme and implicit Runge-Kutta scheme. These numerical schemes are implemented straightforward. Fourdifferent examples based on second order nonlinear second order two-point boundary value problems have been solved and their numerical results have been compare with the exact solution that show the correctness and exactness of the both numerical Adams nscheme and implicit Runge-Kutta schemes. The details of the achieved numerical results in the form of tables as well as figures have been discussed.

**Keywords:** two-point boundary value problems, Adams scheme, implicit Runge-Kutta method, nonlinear, exact solutions

**RESUMEN**

El presente studio es relevante para resolver problemas de valor de límite de dos puntos de segundo orden no lineal de segundo orden mediante el uso de esquemas numéricos rápidos, rápidos y eficientes denominados esquema numérico Adams y esquema implícito Runge-Kutta.
Estos esquemas numéricos se implementan directamente. Se han resuelto cuatro ejemplo diferentes basados en problemas de valor de límite de dos puntos de segundo orden no lineal de segundo orden y sus resultados numéricos se han comparado con la solución exacta que muestras la exactitud de los esquemas numéricos Adams nscheme y Runge-Kutta implícitos. Se han discutido los detalles de los resultados numéricos logrados en forma de tablas y figuras

**PALABRAS CLAVES:**Problemas de valor límite de dos puntos, esquema de Adams, método implícito de Runge Kutta, soluciones no lineales y exactas

## 1 Introduction

Recently, the two-point boundary value problems (TP-BVPs) have been discussed in numerous applications of engineering, science and technology [1, 2]. These TP-BVPs arise in many physical models, for example, transfer of heat, chemical reaction models, fluid viscosity, reaction diffusions, defection of beams and the optimal control problem solutions, etc. Due to the extensive applications and significance of TP-BVPs in the area of science, technology and engineering, it is necessary to find the solutions of such global worth problems. Hence many researcher works to solve these problems by their own way and useful techniques, some of them are Adomian decomposition (AD) method, [3–7], extended (AD) method [8], differential transformation scheme [9], variational iteration technique [10], perturbation methods [11–13] and many more. Perturbation techniques are not difficult to

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solvebut they need small parameter values, which are sometimes considered a difficult task.In recent years, V. Marinca et al. accessible optimalhomotopy asymptotic scheme [14] for for solving the TP-BVPs that don't need of the small parameters. TEscheme is also applied for solving the stationary results of somepartial differential equations, e.g., nonlinear parabolic problems, gKdvtype of equations and many more [15–20]. In optimal homotopy asymptotic scheme, the idea of homotopy technique is applied together with the perturbation

schemes. Here, optimal homotopy asymptotic scheme is used to TP-BVPs to check theapplicability of optimal homotopy asymptotic scheme for TP-BVP. The general form of TP-BVPs is written as:

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} = f(t,y), \qquad a < t < b$$
$$u(a) = A,\ u(b) = B. \tag{1}$$

The rest parts of the paperis organized as follows: Section 2 narrates the numerical technique, the numerical outcomes are provided in Section 3 and conclusions along with future research directions are listed in the final Section.

## 2 Numerical methods

In this study, predictor-corrector numerical Adams methodalong with numerical Implicit Runge-Kutta scheme are applied for solving the two points boundary value problems

### 2.1 Predictor-corrector Adams numerical scheme

To solve and find the numerical measures of the two points boundary value problems. The formulation of corrector-predictor numerical technique is used that takes two steps to complete further.

Step 1: The approximate results for prediction analysis is accomplished

Step 2: To find the numerical values of correction is capable with the same donations of prediction.

$$\frac{du}{dt} = f(t,u), u(t_0) = u_0 \tag{2}$$

For the generalized Adams-Bashforth numerical scheme by using the predictor-corrector is given as:

$$D_{n+1} = u_n + \frac{3}{2} fh(t_n,u_n) - \frac{1}{2} fh(t_{n-1},u_{n-1}) \tag{3}$$

Adams-Moulton two-steps corrector is given as:

$$u_{n+1} = u_n + \frac{1}{2} f\left(h(t_{n+1},D_{n+1}) + h(t_n,u_n)\right). \tag{4}$$

The 4-stepspredictorcorrector scheme is as follows:

$$D_{n+1} = u_n + \frac{f}{24}\left(55h(t_n,u_n) - 59f(t_{n-1},u_{n-1}) + 37f(t_{n-2},u_{n-2}) - 9f(t_{n-3},u_{n-3})\right). \tag{5}$$

The 4-step Adams-Bashforth Moulton method is given as:

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$$u_{n+1} = u_n + \frac{f}{24}\left(9h\left(t_{n+1}, D_{n+1}\right) + 19f\left(t_n, u_n\right) - 5f\left(t_{n-1}, u_{n-1}\right) + f\left(tx_{n-2}, u_{n-2}\right)\right). \quad (6)$$

### 2.2 Implicit Runge-Kutta numerical technique

To solve the two-point boundary value problems, the Implicit Runge-Kutta scheme is performed. The generic form of Implicit Runge-Kutta scheme is narrates as:

$$u_{n+1} = u_n + f\sum_{j=1}^{s} b_j k_j, k_1 = h\left(t_n, u_n\right),$$

$$k_2 = h\left(x_n + c_2 f, u_n + f\left(a_{21}k_1\right)\right),$$

$$k_3 = h\left(x_n + c_3 f, u_n + f\left(a_{31}k_1 + a_{32}k_2\right)\right), \quad \ldots. \quad (7)$$

$$k_s = h\left(x_n + c_s f, u_n + f\left(a_{s1}k_1 + a_{s2}k_2 + \ldots + a_{ss-1}k_{s-1}\right)\right).$$

In the first phase, to consider the attained initial results and slopes are recognized for all the variables. Taking these accomplished numerical values for slopes at the central point of the interval using the designs of dependent variable. In the second phase, the middle-point slopes are achieved by using these accomplished mid-point values. The designed numerical measures for slopes are warped back to the first point to change the other set of middle point results that are invent to the new slope of predictions at middle-point. These numerical measures are additionallyfunctional to create predictions to development slopes at the end point of the interval, Similarly, all the results are obtained to compose the another set of growth functions. Lastly, take at the initial point to make the concluding prediction.

## 3 Results and discussions

This section shows the numerical investigations for nonlinear TW-BVPs with stiff/non-stiff limitations. Some problem have been solved and their numerical measures are tabulated in Tables.

Problem 1: Consider the nonlinear two-point boundary value problem is:

$$\frac{d^2u}{dt^2} = \frac{1}{2}(t+u+1)^3,\ 0 < t < 1$$
$$u(0) = 0,\ u(1) = 0. \quad (8)$$

The exact solution of the above equation is $\frac{2}{2-t} - t - 1$.

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Table 1: Comparison of Exact and numerical solutions for Problem 1

| t | Exact | Adams | Implicit RK |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0.05 | -0.0244 | 0.0243 | 0.0243 |
| 0.1 | -0.0474 | -0.0473 | -0.0473 |
| 0.15 | -0.0689 | -0.0689 | -0.0689 |
| 0.2 | -0.0889 | -0.0888 | -0.0888 |
| 0.25 | -0.1071 | -0.1071 | -0.1071 |
| 0.3 | -0.1235 | -0.1235 | -0.1235 |
| 0.35 | -0.1379 | -0.1378 | -0.1378 |
| 0.4 | -0.15 | -0.1499 | -0.1499 |
| 0.45 | -0.1597 | -0.1596 | -0.1596 |
| 0.5 | -0.1667 | -0.1666 | -0.1666 |
| 0.55 | -0.1707 | -0.1706 | -0.1706 |
| 0.6 | -0.1714 | -0.1714 | -0.1714 |
| 0.65 | -0.1685 | -0.1685 | -0.1685 |
| 0.7 | -0.1615 | -0.1615 | -0.1615 |
| 0.75 | -0.15 | -0.1499 | -0.1499 |
| 0.8 | -0.1333 | -0.1333 | -0.1333 |
| 0.85 | -0.1109 | -0.1108 | -0.1108 |
| 0.9 | -0.0818 | -0.0818 | -0.0818 |
| 0.95 | -0.0452 | -0.0452 | -0.0452 |
| 1 | 0 | 0 | 0 |

The graphs of absolute error of the exact and numerical solutions are presented in Figure 1. It is clear that the error is closed to 10-07 that is considered very good. This absolute error shows the correctness and exactness of the proposed scheme.

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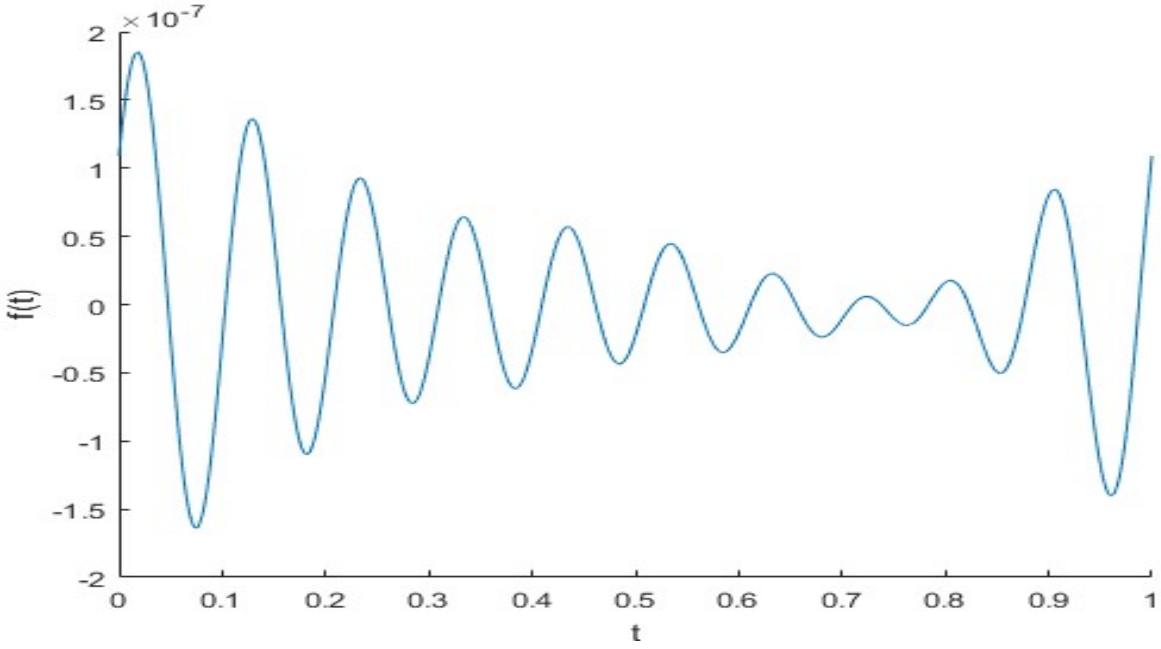


Figure1: Absolute error based on exact solutions and numerical outcomes for Problem 1

Problem 2: Consider the highly nonlinear two-point boundary value problem is:

$$\frac{d^2u}{dt^2} = u^3 - u\frac{du}{dt},\ 0 < t < 1$$
$$u(1) = \frac{1}{2},\ u(2) = \frac{1}{3}. \tag{8}$$

The exact solution of the above equation is $\frac{1}{1+t}$

It is clear in table 2 that the exact and numerical solutions based on Admas and Implicit RK clearly same up to 7 to 8 decimal place. This clearly indicate the performance of the both numerical performances and enhance the worth of the designed scheme.

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Table 2: Comparison of Exact and numerical solutions for Problem 2

| t | Exact | Adams | Implicit RK |
|---|---|---|---|
| 0 | 1.0000 | 1.0000 | 1.0000 |
| 0.05 | 0.9524 | 0.9524 | 0.9524 |
| 0.1 | 0.9091 | 0.9091 | 0.9091 |
| 0.15 | 0.8696 | 0.8696 | 0.8696 |
| 0.2 | 0.8333 | 0.8333 | 0.8333 |
| 0.25 | 0.8000 | 0.8000 | 0.8000 |
| 0.3 | 0.7692 | 0.7692 | 0.7692 |
| 0.35 | 0.7407 | 0.7407 | 0.7407 |
| 0.4 | 0.7143 | 0.7143 | 0.7143 |
| 0.45 | 0.6897 | 0.6897 | 0.6897 |
| 0.5 | 0.6667 | 0.6667 | 0.6667 |
| 0.55 | 0.6452 | 0.6452 | 0.6452 |
| 0.6 | 0.6250 | 0.6250 | 0.6250 |
| 0.65 | 0.6061 | 0.6061 | 0.6061 |
| 0.7 | 0.5882 | 0.5882 | 0.5882 |
| 0.75 | 0.5714 | 0.5714 | 0.5714 |
| 0.8 | 0.5556 | 0.5556 | 0.5556 |
| 0.85 | 0.5405 | 0.5405 | 0.5405 |
| 0.9 | 0.5263 | 0.5263 | 0.5263 |
| 0.95 | 0.5128 | 0.5128 | 0.5128 |
| 1 | 0.5000 | 0.5000 | 0.5000 |

The graphs of absolute error of the exact and numerical solutions are presented in Figure 2. It is clear that the error is closed to 10-07 that is considered very accurate. This absolute error shows the correctness and exactness of the proposed scheme.

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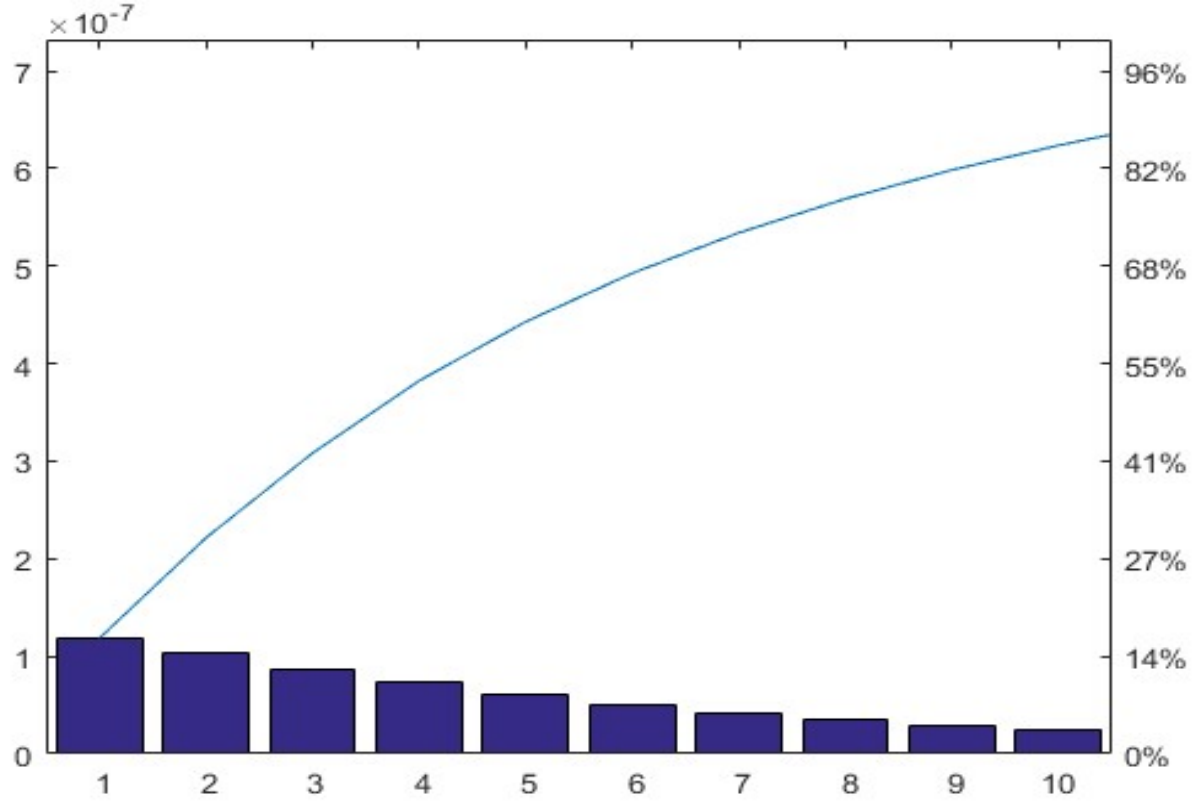


Figure 2: Absolute error based on exact solutions and numerical outcomes for Problem 2

Problem 3: Consider the highly nonlinear two-point boundary value problem is:

$$\frac{d^2u}{dt^2} = \frac{du}{dt} - e^{u-1} - 1,\ 0 < t < 1$$
$$u(0) = 0,\ u(1) = 0. \tag{8}$$

The exact solution of the above equation is $t(1-e^{t-1})$

It is clear in table 3 that the exact/true solutions and the proposed numerical results based on Admas and Implicit RKoverlapped up to 7 to 8 decimal place.

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Table 3: Comparison of Exact and numerical solutions for Problem 3

| t | Exact | Adams | Implicit RK |
|---|---|---|---|
| 0 | 0.0000 | 0.0000 | 0.0000 |
| 0.05 | 0.0307 | 0.0307 | 0.0307 |
| 0.1 | 0.0593 | 0.0593 | 0.0593 |
| 0.15 | 0.0859 | 0.0859 | 0.0859 |
| 0.2 | 0.1101 | 0.1101 | 0.1101 |
| 0.25 | 0.1319 | 0.1319 | 0.1319 |
| 0.3 | 0.151 | 0.1510 | 0.1510 |
| 0.35 | 0.1673 | 0.1673 | 0.1673 |
| 0.4 | 0.1805 | 0.1805 | 0.1805 |
| 0.45 | 0.1904 | 0.1904 | 0.1904 |
| 0.5 | 0.1967 | 0.1967 | 0.1967 |
| 0.55 | 0.1993 | 0.1993 | 0.1993 |
| 0.6 | 0.1978 | 0.1978 | 0.1978 |
| 0.65 | 0.192 | 0.1920 | 0.1920 |
| 0.7 | 0.1814 | 0.1814 | 0.1814 |
| 0.75 | 0.1659 | 0.1659 | 0.1659 |
| 0.8 | 0.145 | 0.1450 | 0.1450 |
| 0.85 | 0.1184 | 0.1184 | 0.1184 |
| 0.9 | 0.0856 | 0.0856 | 0.0856 |
| 0.95 | 0.0463 | 0.0463 | 0.0463 |
| 1 | 0.0000 | 0.0000 | 0.0000 |

The graphs of absolute error of the exact and numerical solutions are presented in Figure 3. It is clear that the error is closed to 10-07 that is considered very accurate. This absolute error shows the correctness and exactness of the proposed scheme.

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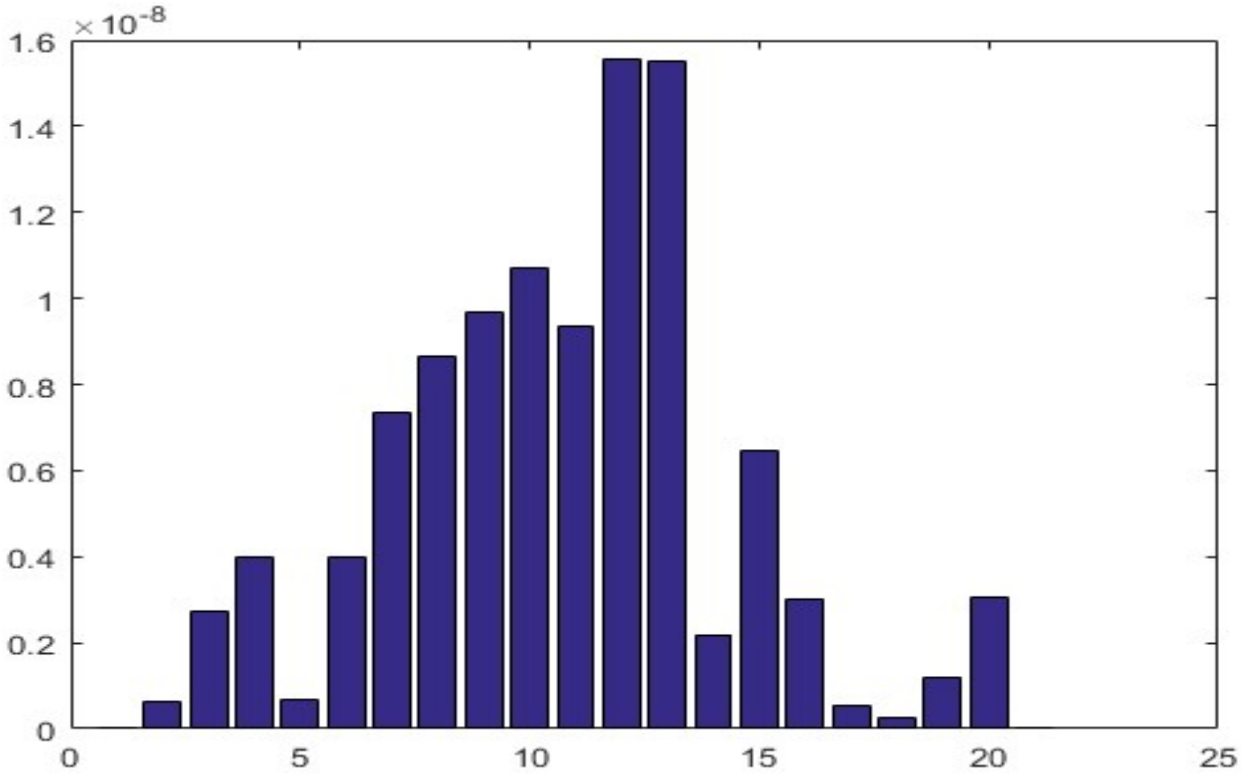


Figure 3: Absolute error based on exact solutions and numerical outcomes for Problem 3

Problem 4: Consider the highly nonlinear two-point boundary value problem having trigonometric functions is given as:

$$\frac{d^2u}{dt^2} = u^2 + 2\pi^2\cos(2\pi t) - \sin^2(2\pi t),\ 0 < t < 1$$
$$u(0) = 0,\ u(1) = 0. \tag{8}$$

The exact solution of the above equation is $\sin^2(\pi t)$

It is clear in table 4 that the exact/true solutions and the proposed numerical results based on Admas and Implicit RK overlapped up to 4 to 5 decimal place.

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Table 4: Comparison of Exact and numerical solutions for Problem 4

| t | Exact | Adams | Implicit RK |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0.05 | 0.0245 | 0.0276 | 0.0276 |
| 0.1 | 0.0955 | 0.1015 | 0.1015 |
| 0.15 | 0.2061 | 0.2141 | 0.2141 |
| 0.2 | 0.3455 | 0.354 | 0.354 |
| 0.25 | 0.5 | 0.5071 | 0.5071 |
| 0.3 | 0.6545 | 0.6585 | 0.6585 |
| 0.35 | 0.7939 | 0.7935 | 0.7935 |
| 0.4 | 0.9045 | 0.8997 | 0.8997 |
| 0.45 | 0.9755 | 0.9675 | 0.9675 |
| 0.5 | 1 | 0.9908 | 0.9908 |
| 0.55 | 0.9755 | 0.9675 | 0.9675 |
| 0.6 | 0.9045 | 0.8997 | 0.8997 |
| 0.65 | 0.7939 | 0.7935 | 0.7935 |
| 0.7 | 0.6545 | 0.6585 | 0.6585 |
| 0.75 | 0.5 | 0.5071 | 0.5071 |
| 0.8 | 0.3455 | 0.354 | 0.354 |
| 0.85 | 0.2061 | 0.2141 | 0.2141 |
| 0.9 | 0.0955 | 0.1015 | 0.1015 |
| 0.95 | 0.0245 | 0.0276 | 0.0276 |
| 1 | 0 | 0 | 0 |

The graphs of absolute error of the exact and numerical solutions are presented in Figure 3. It is clear that the error is closed to 10-03 that is considered very accurate. This absolute error shows the correctness and exactness of the proposed scheme.

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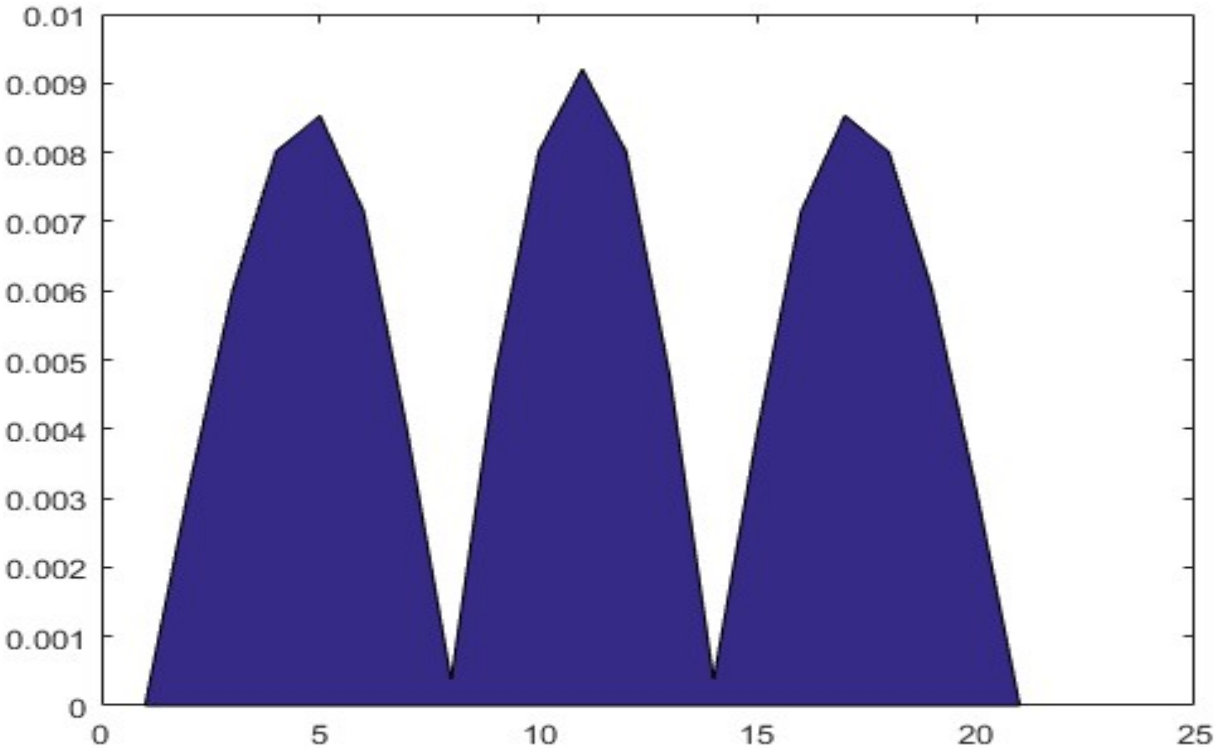


Figure 4: Absolute error based on exact solutions and numerical outcomes for Problem 4

**4Conclusion**

In the present study, two numerical schemes based on Implicit Runge-Kutta scheme and Adams numerical methods have been used for solving a class of nonlinear model using two-point boundary value problems.These numerical treatments have been used for solving four different nonlinear problems of stiff/ non-stiff nature. The numerical investigations have been drawn in tables for each problem and comparison of results has been provided with the exact solutions up to 6 to 7 decimal places. The absolute errors in the form of figures have been plotted for each problem. The results of absolute error indicates that the values are found to be very accurate and the exactness of the designed scheme. The software used for solving the nonlinear second order Van der Pol oscillator is Mathematica 10.4.
In future, this numerical technique will be applied for solving the two-point nonlinear system of differential equations.

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