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TEMPORAL COHOMOLOGY ON NON-STATIONARY MANIFOLDS

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Abstract
Many scientific and engineering systems are naturally modeled as shapes or state spaces that evolve over time. Classical cohomology assigns invariants to a fixed manifold, while per- sistent homology analyzes a filtration parameter that is typically not literal time and usually requires nested inclusions. We introduce *temporal cohomology*, a cohomological framework for time-indexed manifolds that explicitly encodes both spatial and temporal variation and supports non-monotone evolution. We give formal definitions, prove an invariance result under smooth isotopy, and provide a discretization strategy suitable for computations on time-sampled tri- angulations. We compare the construction to persistent homology and discuss applications in medical imaging, climate and Earth systems, robotics and motion planning, evolving networks, and materials science.

1 INTRODUCTION

Topology provides invariants that capture qualitative structure such as connected components, tunnels, and voids. In many modern settings, however, the relevant object is not static: anatomical structures deform during respiration and cardiac cycles, landforms and hydrological networks evolve over seasons, robot configuration spaces change as obstacles move, and interaction networks rewire over time. A central question in these contexts is not only *what* topological features exist, but *when* they exist and *how* they evolve.

Persistent homology summarizes the lifetimes of topological features across a filtration parame- ter (often a scale, threshold, or density level) and yields barcodes or persistence diagrams [1, 2, 3]. While powerful, the classical persistence setup is most natural for nested sequences of spaces. Time evolution, in contrast, can be non-monotone: features may appear, disappear, and reappear, and consecutive time slices need not be related by inclusion.

This paper proposes a cohomological construction that treats time as an intrinsic direction. Informally, we build a differential complex on "space–time forms" and define temporal cohomology as the resulting cohomology groups. When the manifold evolves smoothly without topological change (via isotopy), temporal cohomology reduces to classical cohomology; when topology changes, temporal cohomology records these events through the behavior of temporal classes.

Contributions.

- We define *non-stationary manifolds* and *temporal differential forms* and introduce a *temporal exterior derivative* that combines spatial and temporal variation.
- We define *temporal cohomology* groups and prove invariance under smooth isotopy.
- We give a discretization and algorithmic sketch for time-sampled triangulations, emphasizing a transport-based finite difference for the temporal term.
- We compare temporal cohomology to persistent homology (and note relations to zigzag persis- tence [4]).
- We discuss five application domains where time-evolving topology is central.

2 NON-STATIONARY MANIFOLDS AND TEMPORAL FORMS

2.1 Time-indexed manifolds

Definition 2.1 (Time-indexed manifold family)**.** *Let $T \subset \mathrm{R}$ be an interval. A* time-indexed family of manifolds *is a family $\{M_t\}_{t\in T}$ where each M_t is a smooth n-manifold (possibly with boundary).*

To avoid pathological changes, we assume the family arises from a smooth embedding into an ambient space.

Definition 2.2 (Non-stationary manifold (embedded model))**.** *A* non-stationary manifold *over T is a smooth map*

$$\Phi : M \times T \longrightarrow \mathrm{R}^N$$

such that, for each $t \in T$, the map $\Phi_t := \Phi(\cdot, t)$ is an embedding and $M_t := \Phi_t(M)$ is the observed manifold at

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time t.

This model captures deformation through time. Topological transitions (e.g. handle attach- ments) can be accommodated by allowing Φ_t to lose regularity at isolated times; in this paper we focus primarily on the smooth regime and treat transitions as events detectable in data.

2.2 Temporal differential forms

A key technical issue is that $\Omega^k(M_t)$ changes with t. We handle this by transporting forms back to a fixed reference M.

Let $\Phi_t : M \to M_t$ be a diffeomorphism onto its image. Define the pullback identification

$$\Omega^k(M_t) \xrightarrow{\Phi_t^*} \Omega^k(M).$$

A *temporal k-form* will be represented as a smooth curve in $\Omega^k(M)$ via these identifications.

Definition 2.3 (Temporal differential form)**.** *A* temporal k-form *is a smooth map*

$$\omega : T \to \Omega^k(M), \qquad t \mapsto \omega(t),$$

interpreted as the family $\{\tilde{\omega}(t) \in \Omega^k(M_t)\}$ *with* $\omega(t) = \Phi_t^* \tilde{\omega}(t)$. *We write* Ω^k_T *for the space of temporal k-forms.*

Definition 2.4 (Temporal exterior derivative)**.** *Define* $d_T : \Omega^k_T \to \Omega^{k+1}_T \oplus \Omega^k_T \, dt$ *by*

$$d_T \omega := d\omega + (\partial_t \omega) \wedge dt,$$

where d is the exterior derivative on M and ∂_t *is the time derivative taken in* $\Omega^k(M)$.

This resembles the exterior derivative on the product manifold $M \times T$ written in split form.

Proposition 2.1. *For temporal forms,* $d_T \circ d_T = 0$.

Proof. Compute using $d^2 = 0$, ∂^2 commutativity, and $dt \wedge dt = 0$:

$$d_T(d_T \omega) = d(d\omega) + d(\partial_t \omega) \wedge dt + \partial_t(d\omega) \wedge dt + \partial_t(\partial_t \omega)\, dt \wedge dt = 0.$$

□

3 Temporal cohomology

Definition 3.1 (Temporal cohomology)**.** *Define*

$$H^k_T(M) := \frac{\ker(d_T : \Omega^k_T \to \Omega^{k+1}_T \oplus \Omega^k_T \, dt)}{\mathrm{im}(d_T : \Omega^{k-1}_T \to \Omega^k_T \oplus \Omega^{k-1}_T \, dt)}.$$

Remark 3.1 (Interpretation)**.** *A temporal cocycle is a temporal form whose spatial differential is balanced by its time variation; temporal exactness identifies classes differing by a temporal coboundary. Intuitively,* H^k_T *captures cohomological structure of the evolving system in a way that is sensitive to time dependence rather than solely snapshot topology.*

3.1 Motivating schematic example

Figure 1 illustrates a "rubber donut" whose central tunnel exists only during part of the evolution. Temporal cohomology is designed to distinguish deformation (no topological change) from eventful topology changes.

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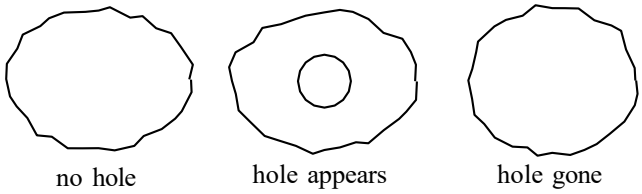

Figure 1: Schematic "birth" and "death" of a hole during evolution. A temporal class exists over the interval where the hole is present.

4 INVARIANCE UNDER SMOOTH ISOTOPY

A minimum correctness requirement is that if the manifold only deforms smoothly (no topology change), the theory should reduce to classical cohomology.

Definition 4.1 (Smooth isotopy regime)**.** *The family $\{M_t\}$ is generated by a smooth isotopy if there exist diffeomorphisms $\psi_{t\leftarrow t0} : M_{t0} \to M_t$ varying smoothly with t.*

Theorem 4.1 (Temporal invariance under isotopy)**.** *If $\{M_t\}$ is generated by a smooth isotopy, then for each k there is a natural isomorphism*

$$H^k_T(M) \cong H^k(M_{t_0})$$

for any fixed $t_0 \in T$.

Proof sketch. Under isotopy, all slices M_t are diffeomorphic to M_{t0}, so classical cohomology is invariant: $H^k(M_t) \cong H^k(M_{t_0})$ for all t. Using the identification Φ^* (or $\psi^*_{t\leftarrow t_0}$), any temporal form $\omega(t)$ corresponds to a smooth curve in $\Omega^k(M_{t_0})$. Temporal closedness reduces to spatial closedness (up to temporal exact adjustments), and temporal exactness reduces to classical exactness under the same identification. One then constructs an explicit inverse using a choice of representative and shows the mapping is well-defined on equivalence classes. □

Remark 4.1. *Theorem 4.1 is the analogue of a "sanity check": smooth deformation should not create spurious topological signal.*

5 COMPARISON TO PERSISTENT HOMOLOGY

Persistent homology studies a filtration $X_{a0} \subseteq X_{a1} \subseteq \cdots$ and tracks births and deaths of homologi- cal features [1, 2]. Its algebraic object is a persistence module, decomposing into interval summands under tameness assumptions [3]. Temporal cohomology differs in three practical respects:

1. **Time vs. filtration.** The parameter is literal time. Consecutive slices need not be nested; evolution can be non-monotone.
2. **Geometry-aware temporal term.** The temporal derivative couples adjacent times via trans- port; in data settings this captures "how" a feature moves or deforms, not only whether it exists.
3. **Relationship to zigzag persistence.** Discrete-time families without inclusions can be encoded as zigzags [4]. Temporal cohomology offers a smooth-time viewpoint and naturally accommo- dates transport/registration.

Figure 2 summarizes the conceptual distinction.

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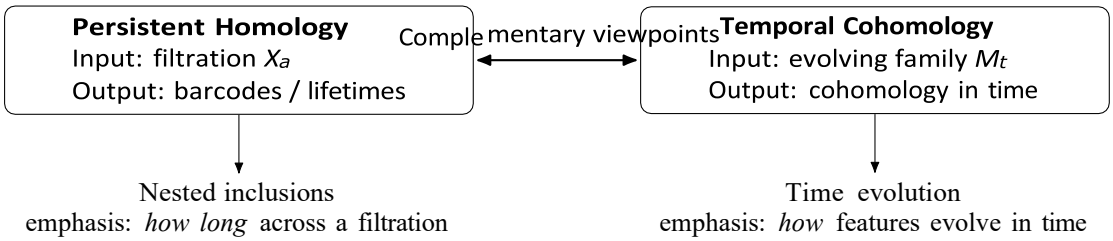

Figure 2: Persistent homology vs. temporal cohomology: filtration lifetimes versus time evolution.

6 Discretization and algorithmic sketch

Many applications provide time-sampled data: meshes or triangulated manifolds $\{K_i\}_{i=0}^{N}$ approximating M_{t_i}. Temporal cohomology can be approximated on cochains by a coupled spatial–temporal coboundary.

6.1 Discrete objects

Let $C^k(K_i;\mathrm{F})$ denote k-cochains over a field F (often Z_2) and let $\delta^k : C^k(K_i) \to C^{k+1}(K_i)$ be the coboundary operator.

A central ingredient is a *transport* map $\tau_i : C^k(K_i) \to C^k(K_{i+1})$ induced by registration or correspondence between consecutive time slices (e.g. via vertex matching, optical flow, or known simulation correspondences).

6.2 Temporal coboundary

Define the discrete temporal difference

$$\Delta^k := \tau_i - I : \; C^k(K_i) \to C^k(K_{i+1}).$$

Build a block operator δ_T acting on the graded sum of cochains across times, with spatial parts δ_i and coupling parts Δ_i (a discrete analogue of $d + \partial_t(\cdot) \wedge dt$).

Algorithm (sketch).

1. For each i, compute sparse matrices for δ^k_i.
2. Choose or estimate transport maps τ_i and form Δ^k_i.
3. Assemble the block matrix representing δ^k_T.
4. Compute $\dim \ker(\delta^k_T)$ and $\dim \mathrm{im}(\delta^{k-1}_T)$ via sparse linear algebra.
5. Output $\dim H^k_T = \dim \ker(\delta^k_T) - \dim \mathrm{im}(\delta^{k-1}_T)$ along with representative temporal cocycles if desired.

Remark 6.1. *This section is intentionally an algorithmic sketch. A full implementation would specify transport construction and provide complexity analysis and stability under registration error. Cohomology-based computations are often efficient in practice and have been leveraged in persistent cohomology implementations [5].*

7 APPLICATIONS

We briefly indicate how temporal cohomology can be operationalized in five domains.

7.1 Medical imaging

Cardiac and respiratory motion produce 4D data (3D + time). Temporal cohomology can separate robust anatomical cavities (persistent temporal classes) from transient artifacts (short-lived tem- poral classes). Longitudinal imaging of tumors can reveal when necrotic cores form (birth of a void class) and whether cavities persist across treatment windows.

7.2 Climate and Earth systems

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Evolving coastlines, lakes, river deltas, and ice melt ponds undergo topology changes: lakes merge or split (changes in H^0), channels form loops (changes in H^1). Temporal cohomology provides time-stamped summaries of these events in a way that is robust to continuous deformation.

7.3 Robotics and motion planning

A robot's collision-free configuration space changes as obstacles move. Temporal cohomology can track the existence of safe corridors (tunnels) through time, supporting plans that exploit timed openings and avoid transient blockages.

7.4 Network evolution

Dynamic graphs and higher-order simplicial complexes encode social, biological, or digital inter- actions. Temporal cohomology can track persistent communication loops and transient motifs, complementing graph statistics with topological descriptors.

7.5 Materials science

Porous media, foams, and soft materials can develop, merge, or lose pores under stress. Temporal cohomology yields lifetime distributions of pores/voids, distinguishing elastic deformation from structural failure.

8 CONCLUSION

Temporal cohomology offers a cohomological language for evolving spaces that is complementary to persistent homology. It treats time as an intrinsic direction and naturally accommodates non- monotone dynamics. Future work includes stability theory under registration error, explicit treat- ment of topological transitions (surgery events), and integration with multi-parameter persistence and statistical learning pipelines.

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