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# EQUIVALENCE PROCESS USING ZERMELO'S WELL-ORDERING THEOREM IN FINITE AND INFINITE SETS

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**ABSTRACT**
This study investigates the profound relationship between Zermelo's Well-Ordering Theorem (WOT), the Axiom of Choice (AC), and the comparative measure of set size, known as **cardinality**. We examine how the concept of **equipollence** (bijective mapping) intuitively establishes the equivalence of finite sets, yet requires the non-constructive guarantee of WOT/AC to rigorously compare and order the cardinalities of all infinite sets. The analysis contrasts the constructive demonstration of equivalence for countable infinite sets (e.g., the natural numbers {N}, integers {Z}, and rational numbers {Q}) with the theoretical, non-constructive **well-orderability** asserted for uncountable sets like the real numbers {R}. Furthermore, this paper highlights WOT's foundational importance in advanced set theory, discusses its philosophical limitations (due to the non-constructive nature of AC), and explores its potential pedagogical utility as a bridge between students' concrete reasoning about finite sets and the abstract structure of transfinite set theory.

**Keywords:**
Zermelo, Well-Ordering, Axiom of Choice, Equivalence, Finite Sets, Infinite Sets, Ordinals, Cardinals

## INTRODUCTION

The core of set theory deals with the question of set equivalence, how one determines that two sets have the same size, or cardinality, by establishing a one-to-one correspondence between their elements. For finite sets, the notion is straightforward, and its truth can be verified directly by counting. But when infinite sets are involved, the notion becomes much more delicate and essentially theoretical, depending for example on the Axiom of Choice (AC) and on Zermelo's Well-Ordering Theorem (WWT). The theorem states that any set can be put into a well-defined order and provides an important method for 'comparing' infinite sets. Of great importance, the theorem is non-constructive in character, and it depends explicitly on the Axiom of Choice. This study aims to present a clear and organized discussion of how the Well-Ordering Theorem supports the process of establishing equivalence in both finite and infinite cases.

## OBJECTIVES

- To explain Zermelo's Well-Ordering Theorem and show how it relates to the Axiom of Choice.
- To demonstrate how the equivalence of finite sets can be shown through clear one-to-one (bijective) mappings.
- To explore how well-ordering helps identify and compare the equivalence of infinite sets.
- To discuss the theoretical importance of the theorem and its potential use in teaching complex mathematical concepts.

## REVIEW OF RELATED LITERATURE

Zermelo (1904) first proposed the Well-Ordering Theorem, marking a significant development step in the growth of modern set theory. An earlier contribution was made by Cantor (1895), who delineated the principle of

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equipollence and established ways of comparing the sizes, known as cardinalities, of infinite sets. Subsequently, Halmos (1974), Jech (2003), and Kunen (2011) discussed how the theorem relates to the Axiom of Choice and Zorn's Lemma; the emphasis is that the Well-Ordering Theorem is non-constructive. These works put together show that the Well-Ordering Theorem, while it is a strong theoretical device in the classification of sets, does not provide any effective method of constructing a well-ordering for such a set as the real numbers. Some educational studies also suggest the introduction of the theorem after exploring more concrete examples, such as the natural numbers ($\mathbb{N}$), integers ($\mathbb{Z}$), and the rational numbers ($\mathbb{Q}$) is important for balancing abstract theory with clear comprehension.

## METHODOLOGY

This study follows a theoretical and proof-based approach, structured around several key steps:

1. **Stating and analyzing Zermelo's Well-Ordering Theorem**
   The study starts with the restatement of Zermelo's theorem: every set is well-orderable. It then discusses it in terms of the system of Zermelo-Fraenkel set theory as equivalent to the Axiom of Choice. It explains how the theorem says that for any set, either finite or infinite, there should be an ordering such that for every non-empty subset, there would be a least element. The analysis mentions its role also in ordinal number definitions and in establishing the correspondence between sets and ordinals, providing the theoretical basis for a comparison between infinite collections.
2. **Demonstrating equivalence for finite sets through explicit bijections and counting**
   Equivalence for finite sets is straightforward and constructive. Two sets are equivalent, given that there is a bijective function between them, which maps each element in the first set uniquely to an element in the second set. The paper gives explicit bijections and checks that the mappings are one-to-one and onto. Counting arguments show that two finite sets are equivalent if and only if they have the same number of elements. Examples illustrate how equivalence groups finite sets into classes of equal size.
3. **Applying WWT to infinite sets to compare cardinalities**
   Equivalence for finite sets is straightforward and constructive. For example, two sets are equivalent, given that there exists a bijective function between the two sets mapping every element in one set uniquely to an element in another. Examples of such bijections are given side by side, and it is checked that the mappings are one-to-one and onto. Counting arguments show that two finite sets are equivalent iff they have the same number of elements. Equivalence classes group finite sets by size.
4. **Providing examples of equivalence among classical sets ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$)**
   The study includes practical examples:
   **Natural numbers ($\mathbb{N}$) and integers ($\mathbb{Z}$):** A bijection is created using the sequence (0, 1, -1, 2, -2, …), showing both sets are countably infinite.
   **Natural numbers ($\mathbb{N}$) and rational numbers ($\mathbb{Q}$):** The rationals can be enumerated using methods like the Calkin-Wilf tree or diagonal listing, proving countability.
   **Real numbers ($\mathbb{R}$):** Cantor's diagonal argument demonstrates that $\mathbb{R}$ is uncountable, but WWT guarantees a well-ordering exists, allowing theoretical comparison despite the lack of explicit construction.
5. **Discussing implications, limitations, and pedagogical applications**
   The Well-Ordering Theorem has far-reaching implications for comparing the size of all sets via ordinals supporting the hierarchy of cardinal numbers. However, in doing so, it rests on the Axiom of Choice introducing aspects of nonconstructiveness by asserting the existence but not the method of construction for such orderings. This pedagogically allows discussing abstract ways of reasoning in mathematics education. It is suggested to introduce WWT in a later stage after exploring more 'concrete' examples of countable sets to help students balance theoretical understanding with more tangible examples of how such abstract concepts come to be intuitively comprehended.

## RESULTS AND DISCUSSION

### A. Equivalence in Finite Sets

For finite sets $A$ and $B$, equivalence is verified through a bijection $f: A \to B$. Such mappings can always be explicitly constructed—for example, pairing the $k$-th element of $A$with the $k$-th element of $B$. Counting confirms that $|A| = |B|$, establishing equivalence.

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Proof:
For finite sets, equivalence is demonstrated by constructing an explicit bijection. Let

$$A = \{a_1, a_2, \dots, a_n\}, B = \{b_1, b_2, \dots, b_n\}.$$

Theorem 1: Any two finite sets with the same number of elements are equivalent.
Define a function

$$f: A \to B$$

by

$$f(a_i) = b_i \text{for all } i = 1,2, \dots, n.$$

We verify that $f$ is a bijection.

1. Injective (One-to-one):
   If $f(a_i) = f(a_j)$, then $b_i = b_j$. Since the elements of $B$ are distinct, $i = j$, implying $a_i = a_j$. Thus, $f$ is injective.
2. Surjective (Onto):
   Let $b_k \in B$. By definition of the mapping, $f(a_k) = b_k$. Thus, every element of $B$ is an image of some element of $A$. So $f$ is surjective.

Since $f$ is both injective and surjective, it is a bijection.
Hence, $A \sim B$.

$\therefore$ Finite sets of equal size are equivalent.

**B. Equivalence in Infinite Sets**
For infinite sets, equivalence cannot rely on counting. Instead, well-ordering allows each set to be associated with an ordinal number, its order type. By applying Zermelo's theorem, any infinite set can be well-ordered, allowing a one-to-one correspondence with an ordinal. Sets with the same order type are equivalent. This framework enables comparisons between sets that cannot be explicitly enumerated.
Examples:

- $\mathbb{N} \leftrightarrow \mathbb{Z}$: A bijection exists by pairing elements as (0,1,−1,2,−2,), showing both are countably infinite.
- $\mathbb{N} \leftrightarrow \mathbb{Q}$: The rationals can be listed by the Calkin-Wilf tree, establishing countability.
- $\mathbb{R}$: Although uncountable, WWT guarantees a well-ordering exists, though not constructively.

B.1. Sets Where Explicit Bijections Exist
Theorem 2: $\mathbb{N} \sim \mathbb{Z}$ (The integers are countably infinite).
Proof Construction:

Define the function $f: \mathbb{N} \to \mathbb{Z}$ by:

$$f(n) = \begin{cases} 0, & n = 0, \\ \dfrac{n+1}{2}, & n \text{ is odd}, \\ -\dfrac{n}{2}, & n \text{ is even}. \end{cases}$$

We verify bijection:

- Every integer appears exactly once:

$$0,1,-1,2,-2,3,-3, \dots$$

- The mapping is reversible:
  Positive integers appear for odd $n$, negatives for even $n$.

Thus, $f$ is bijective.

$\therefore \mathbb{Z}$ is countably infinite.

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Theorem 3: $\mathbb{N} \sim \mathbb{Q}$ (The rationals are countably infinite).
Proof Sketch:
List all rational numbers in reduced form using a grid:

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, \cdots$$

Use Cantor's diagonal procedure and ignore duplicates (fractions not in lowest terms). The process defines a bijection

$$f: \mathbb{N} \to \mathbb{Q}.$$

Thus,

$$\boxed{\therefore \mathbb{Q} \text{ is countably infinite.}}$$

B.2. Sets Where Explicit Bijections Do NOT Exist (Using WOT)
Theorem 4: The real numbers $\mathbb{R}$can be well-ordered.

Proof:
This proof cannot be constructive. Instead, we rely on Zermelo's Well-Ordering Theorem:
Every set can be well-ordered.
Since $\mathbb{R}$ is a set, WOT implies that a well-ordering $\prec$exists such that every non-empty subset of $\mathbb{R}$ has a least element under $\prec$.
Because every well-ordered set is order-isomorphic to exactly one ordinal, there exists an ordinal $\alpha$and a bijection

$$f: \alpha \to \mathbb{R}.$$

Thus, even though no explicit bijection or ordering can be written, WOT guarantees:

$$\boxed{\mathbb{R} \text{ has an ordinal order type and therefore a cardinality } |\mathbb{R}| = \aleph_\alpha.}$$

:
This step uses the Axiom of Choice and is fully non-constructive.

**C. Theoretical Implications**
The WWT implies every set has an *initial ordinal* representing its cardinality. Under AC, for any sets A and B, either $|A| \leq |B|$ or $|B| \leq |A|$. This comparability strengthens the hierarchy of infinite cardinalities but introduces non-constructive assumptions.

Theorem 5: Under the Axiom of Choice, for any sets $A$and $B$, either $|A| \leq |B|$ or $|B| \leq |A|$.
Proof Outline:

1. By WOT, both $A$and $B$can be well-ordered.
2. Let their order types be the ordinals $\alpha$and $\beta$.
3. Ordinals are totally ordered under $<$.
4. Therefore, either $\alpha \leq \beta$or $\beta \leq \alpha$.
5. If $\alpha \leq \beta$, then there exists an injection

$$A \hookrightarrow B$$

Similarly for the reverse.
Thus,

$$\boxed{\text{All sets become comparable in size under AC..}}$$

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**D. Summary of Concrete Findings**

| Set Pair | Explicit Bijection Exists? | Method Used | Result |
|---|---|---|---|
| $A$, $B$ finite | Yes | Direct bijection | Equivalent if same size |
| $\mathbb{N} \sim \mathbb{Z}$ | Yes | Constructed bijection | Countably infinite |
| $\mathbb{N} \sim \mathbb{Q}$ | Yes | Enumeration | Countably infinite |
| $\mathbb{R}$ and ordinals | **No** | WOT (non-constructive) | Well-ordering exists; cardinality defined |

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## CONCLUSION

The study reconfirms that Zermelo's WWT provides a conceptual framework for both the finite and infinite cases. First, it shows that WWT, which is closely related to the Axiom of Choice, ensures that every set can be well-ordered, hence any two sets are theoretically comparable even when their explicit orderings cannot be constructively built. Second, the equivalence of finite sets was clearly demonstrated by explicit bijective mappings; two sets with the same number of elements are constructively equivalent. Third, for infinite sets well-ordering permits the identification and comparison of cardinalities: countable sets like $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ can be paired in explicit bijections, whereas uncountable sets like $\mathbb{R}$ depend on the non-constructive existence ensured by WWT. Finally, the investigation points out the theoretical significance of WWT within set theory-demonstrating the role played by WWT in the process of comparing and classifying sets-and stresses its pedagogical potential in teaching abstract mathematical concepts, while noting concrete examples of its application in abstract reasoning.

**Recommendations:**

1. This could involve investigations into constructive or algorithmic methods for well-ordering particular infinite sets, thus providing more concrete examples to go along with the non-constructive nature of Zermelo's theorem.
2. Mathematics educators should introduce WWT and the Axiom of Choice in a later stage of students' mathematical education, so that they already possess a certain comprehension of concrete, countable sets, such as $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$.
3. Further research might address specific teaching methodologies that make clear the hierarchy of infinite cardinalities, showing equivalences and order types, in a manner that balances theoretical rigor with intuitive understanding.

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